Jurnal EurekaMatika 11(1) (2023) 21-24

Jurnal EurekaMatika

Journal homepage: https://ejournal.upi.edu/index.php/JEM

A Note on Relative Normal Subgroups

Yudi Mahatma[1], Ibnu Hadi[1], dan Binastya Anggara Sekti[2]*

[1]Program Studi Matematika, Universitas Negeri Jakarta, Indonesia
[2]Program Studi Sistem Informasi Universitas Esa Unggul, Indonesia

*Correspondence author Email: yudi_mahatma@unj.ac.id

A B S T R A K

Let G be finite group and $\alpha \in Aut(G)$. In 2016, Ganjali and Erfanian introduced the notion of a normal subgroup of G which is relative to α, called the α-normal subgroup. In detail, N is an α-normal subgroup of G if for every $g \in G$ and $n \in N$ we have $g^{-1}n\alpha(g) \in N$. In this research, we show that if N is an α-normal subgroup of G and $\tau \in Aut(G)$ then N is τ-normal in G if and only if $\alpha(g)\tau(g^{-1}) \in N$ for every $g \in G$.

INFORMASI ARTIKEL

Sejarah Artikel:
Diterima 10 Februari 2023
Direvisi 20 Maret 2023
Disetujui 2 April 2023
Tersedia online 1 Mei 2023
Dipublikasikan 1 Juni 2023

Keywords:
Group,
Normal Subgroup,
α-normal Subgroup.

A B S T R A C T

Misalkan G grup hingga dan $\alpha \in Aut(G)$. Tahun 2016, Ganjali dan Erfanian memperkenalkan ide tentang suatu subgrup normal dari G relatif terhadap α, dinamakan subgrup normal-α. Secara rinci, N adalah suatu subgrup normal-α dari G apabila untuk setiap $g \in G$ dan $n \in N$ berlaku $g^{-1}n\alpha(g) \in N$. Dalam penelitian ini diperlihatkan bahwa jika N adalah suatu subgrup normal-α dari G dan $\tau \in Aut(G)$ maka N normal-τ di G jika dan hanya jika $\alpha(g)\tau(g^{-1}) \in N$ untuk setiap $g \in G$.

Kata Kunci:
Grup,
Subgrup Normal,
Subgrup Normal-α.

1. INTRODUCTION

Let G be a group. A subgroup N of G is said to be *normal* if for any $g \in G$, $g^{-1}ng \in N$ for every $n \in N$. It was shown in the book of 'Topics in Algebra' chapter 'Groups' by Herstein that the followings are equivalent:

1. The subgroup N is normal in G.
2. For every $g \in G$, $g^{-1}Ng = N$.
3. For every $g \in G$, $gN = Ng$.
4. For every $a, b \in G$, $NaNb = Nab$.

Now suppose that $\alpha \in Aut(G)$ be any automorphism of G. In Khukhro & Makarenko (2007) and Ganjali & Erfanian (2017), it was introduced the concept of the normal subgroup of a finite group G which is related to α, that is, the subgroup N of G satisfying $g^{-1}n\alpha(g) \in N$ for any $g \in G$ and $n \in N$. We call such subgroup as the α-normal subgroup. Specifically, every ordinary normal subgroup can be regarded as 1-normal subgroup, where $1 \in Aut(G)$ denotes the identity map on G.

Further results and wider study based on the α-normal subgroup can be found in Read (1976), Mazur (1994), Barzegar (2015), Kumar (2019), Ganjali & Erfanian (2020), Haghparast *et. al*., (2021), Haghparast *et. al*., (2023). Recently, Mahatma, *et al.*, (2021) formulated the basic properties of such subgroups and obtain some results analogous to the classic version. They showed that if G is a group and $\alpha \in Aut(G)$ then the followings are equivalent:

1. The subgroup N is α-normal in G.
2. For every $g \in G$, $g^{-1}N\alpha(g) = N$.
3. For every $g \in G$, $gN = N\alpha(g)$.
4. For every $a, b \in G$, $NaNb = Na\alpha(b)$.

Now suppose that N is α-normal. Our goal is to find the necessary and sufficient conditions for an automorphism τ so that N is τ-normal.

2. METHODS

Suppose that G is a finite group and N is a subgroup of G. Clearly, if $N = G$ then N will be α-normal in G for any $\alpha \in Aut(G)$. Hence, we will assume that the subgroup N is proper. Now, notice that if $N = \{e\}$ then, for every $n \in N$, $g^{-1}n\alpha(g) \in N$ implies that $\alpha(g) = g$ for every $g \in G$. Thus, the only automorphism that can provide the relative normality of N is the identity mapping. Based on this fact, we restrict the discussion only for the case where the subgroup N is nontrivial.

Suppose that N is an α-normal subgroup of G for some $\alpha \in Aut(G)$. First, we will formulate the necessary condition for an automorphism $\tau \in Aut(G)$ so that N can be regarded as a τ-normal subgroup. Next, we show that such a condition is also sufficient.

3. RESULTS AND DISCUSSION

Let G be a group and N be a nontrivial subgroup of G such that N is α-normal in G for an $\alpha \in Aut(G)$. It was shown in Mahatma, *et al.*, (2021) that, for every $h \in G$, $hN = N\alpha(h)$. Now suppose that $\tau \in Aut(G)$ such that N is τ-normal in G. Thus, for every $h \in G$, we have $hN = N\tau(h)$. Consequently, $N\alpha(h) = N\tau(h)$ for every $h \in G$. This equality holds if and only if $\alpha(h)\big(\tau(h)\big)^{-1} = \alpha(h)\tau(h^{-1}) \in N$. This gives the necessary condition for τ.

Now suppose that $\sigma \in Aut(G)$ satisfies $\alpha(h)\sigma(h^{-1}) \in N$ for every $h \in G$. Let $g \in G$ and $n \in N$. Notice that if $\alpha(g)\sigma(g^{-1}) = n'$ then $\sigma(g^{-1}) = \big(\alpha(g)\big)^{-1}n' = \alpha(g^{-1})n'$ and hence

$gn\sigma(g^{-1}) = gn\alpha(g^{-1})n'$. Since N is α-normal then $gn\alpha(g^{-1}) \in N$ and thus, $gn\sigma(g^{-1}) \in N$. This shows that N is α-normal. We summarize the result in the following theorem:

Theorem 1 *Let G be finite group and N be subgroup of G. Suppose that N is α-normal in G for an $\alpha \in Aut(G)$. If $\tau \in Aut(G)$ then N is τ-normal in G if and only if $\alpha(g)\tau(g^{-1}) \in N$ for every $g \in G$.*

As an example, let N be α-normal subgroup of G and let $n \in N$. Consider the inner automorphism $\tau_n \in Aut(G)$ defined by $\tau_n(x) \coloneqq n^{-1}xn$ for every $x \in G$. Let $g \in G$. Write $y = \alpha(g)\tau_n(g^{-1}) = \alpha(g)n^{-1}g^{-1}n$. Since N is α-normal then $y^{-1} = (\alpha(g)n^{-1}g^{-1}n)^{-1} = n^{-1}gn\alpha(g^{-1}) \in N$ and thus, $y \in N$. According to Theorem 1, N is τ_n-normal.

Remark

The example above shows that once a subgroup N of G is α-normal for an $\alpha \in Aut(G)$, then N is τ_n-normal for every $n \in N$. Now, since $\tau_e = 1$, the identity mapping on G, then N is 1-normal which means that N is an ordinary normal subgroup of G.

Let us continue with further investigation. Suppose that the subgroup N is both α-normal and τ-normal in G. It is clear from the definition that, for every $g \in G$, we have $g^{-1}\alpha(g) \in N$ and $g^{-1}\tau(g) \in N$. Now let $g \in G$ and write $g^{-1}\tau(g) = n'$. Thus, $\tau(g) = gn'$ and hence, for every $n \in N$, $g^{-1}n\alpha\tau(g) = g^{-1}n\alpha(gn') = g^{-1}n\alpha(g)\alpha(n')$. Now, it was shown in Mahatma, *et al.*, (2021) that α must satisfy $\alpha(N) = N$ and thus, $\alpha(n') \in N$. Next, since N is α-normal then $g^{-1}n\alpha(g) \in N$ and thus, we have $g^{-1}n\alpha\tau(g) \in N$. Since this relation holds for any $g \in G$ and $n \in N$ then we conclude that N is $\alpha\tau$-normal.

Now suppose that $\beta \in Aut(G)$ be the inverse of α i.e., $\alpha\beta = \beta\alpha = 1$. Let $g \in G$. Since N is α-normal then we have $\left(\beta(g)\right)^{-1}n^{-1}g = \left(\beta(g)\right)^{-1}n^{-1}\alpha\beta(g) \in N$ for every $n \in N$. Consequently, we have $g^{-1}n\beta(g) \in N$. Since this relation holds for any $g \in G$ and $n \in N$ then we conclude that N is β-normal.

We summarize this result in the following theorem:

Theorem 2 *Let G be finite group and N be normal subgroup of G. Then the set* $\mathrm{H} \coloneqq \{\tau \in \mathrm{Aut(G)} | \mathrm{N\ is\ \tau\text{-}normal\ in\ G}\}$ *is a subgroup of $Aut(G)$.*

4. CONCLUSION

We have seen in the discussion that if N is an α-normal subgroup of G and $\tau \in Aut(G)$ then N is τ-normal in G if and only if $\alpha(g)\sigma(g^{-1}) \in N$ for every $g \in G$. Moreover, if N is an α-normal subgroup then N is τ_n-normal for all $n \in N$ where τ_n is inner automorphism of G. But this implies that every α-normal subgroup must be an ordinary normal subgroup.

5. ACKNOWLEDGEMENT

The This research is fully funded and supported by Lembaga Penelitian dan Pengabdian Masyarakat Universitas Negeri Jakarta with contract number 41 /SPK PENELITIAN/5.FMIPA/2021.

6. REFERENCES

Barzegar, R. (2015). Nilpotency and solubility of groups relative to an automorphism. *Caspian Journal of Mathematical Sciences (CJMS)*, *4*(2), 271-283.

Ganjali, M., & Erfanian, A. (2017). Perfect groups and normal subgroups related to an automorphism. *Ricerche di Matematica, 66*(2), 407-413.

Ganjali, M., & Erfanian, A. (2020). Some notes on relative commutators. *Indonesian Journal of Pure and Applied Mathematics*, *2*(2), 65-70.

Haghparast, M., Moghaddam, M. R. R., & Rostamyari, M. A. (2021). Some results of left and right α-commutators of groups. *Asian-European Journal of Mathematics, 14*(1), 2050148.

Haghparast, M., R Moghaddam, M. R., & Rostamyari, M. A. (2023). Some results of α -coset groups. *Algebraic Structures and Their Applications*, *10*(1), 87-93.

Khukhro, E. I., & Makarenko, N. Y. (2007). Large characteristic subgroups satisfying multilinear commutator identities. *Journal of the London Mathematical Society*, *75*(3), 635-646.

Kumar, P. (2019). On commuting automorphisms of finite groups. *Ricerche di Matematica*, *68*(2), 899-904.

Mahatma, Y., & Hadi, I. (2021). Relation between the left and right cosets of an α-normal subgroup. *Journal of Physics: Conference Series, 2106*(1), 012023.

Mazur, M. (1994). Automorphisms of finite groups. *Communications in Algebra*, *22*(15), 6259-6271.

Read, E. W. (1976). The α-regular classes of the generalized symmetric group. *Glasgow Mathematical Journal*, *17*(2), 144-150.