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Third Version of Weak Orlicz–Morrey Spaces and Its Inclusion Properties

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ABSTRACTS

Orlicz–Morrey spaces are generalizations of Orlicz spaces and Morrey spaces which were first introduced by Nakai. There are three versions of Orlicz–Morrey spaces. In this article, we discussed the third version of weak Orlicz –Morrey space, which is an enlargement of third version of (strong) Orlicz – Morrey space. Similar to its first version and second version, the third version of weak Orlicz -Morrey space is considered as a generalization of weak Orlicz spaces, weak Morrey spaces, and generalized weak Morrey spaces. This study investigated some properties of the third version of weak Orlicz –Morrey spaces, especially the sufficient and necessary conditions for inclusion relations between two these spaces. One of the keys to get our result is to estimate the quasi- norm of characteristics function of open balls in $\mathbb{R}^n$.

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1. INTRODUCTION

Orlicz-Morrey spaces are generalization of Orlicz spaces and Morrey spaces and it is firstly introduced by E. Nakai in 2004 (Maeda *et al*., 2013 ; Nakai , 2008 ; Deringoz *et al*., 2015). These paces are one of the important topics in mathematical analysis , particularly in harmonic analysis . There are three versions of Orlicz –Morrey spaces , i.e: Nakai ’s, Sawano –Sugano –Tanaka ’s (Gala *et al*., 2015) and Deringoz –Guliyev –Samko ’s (Deringoz *et al*., 2015) versions.

For a Young function $\Theta: [0,\infty) \to [0,\infty)$ (i.e. Θ is convex, $\lim_{t\to 0}\Theta(t) = 0 = \Theta(0)$, continuous and $\lim_{t\to\infty}\Theta(t) = \infty$), we define $\Theta^{-1}(s) := \inf\{r \geq 0: \Theta(r) > s\}$. Given two Young functions Θ_1, Θ_2, we write $\Theta_1 \prec \Theta_2$ if there exists a constant $C > 0$ such that $\Theta_1(t) \leq \Theta_2(Ct)$ for all $t > 0$.

Now, let G_θ be the set of all functions $\theta: (0,\infty) \to (0,\infty)$ such that $\theta(r)$ is decreasing but $\Theta^{-1}(t^{-n})\theta(t)^{-1}$ is almost decreasing for all $t > 0$. Let $\theta_1 \in G_{\theta_1}$ and $\theta_2 \in G_{\theta_2}$, we denote $\theta_1 \lesssim \theta_2$ if there exists a constant $C > 0$ such that $\theta_1(t) \leq C\theta_2(t)$ for all $t > 0$.

First we recalled definition of (strong) Orlicz–Morrey spaces of Deringoz–Guliyev–Samko’s (2015) version. Let Θ be a Young function and $\theta \in G_\theta$, *the Orlicz–Morrey space* $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ is the set of measurable functions f on $\mathbb{R}^n$ such that

$$\| f \|_{\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n, r>0} \frac{\Theta^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta\left(|B(a,r)|^{\frac{1}{n}}\right)} \| f \|_{L_\Theta(B)} < \infty,$$

where $\| f \|_{L_\Theta(B)} := \inf\{b > 0: \int_{B(a,r)} \Theta\left(\frac{|f(x)|}{b}\right) dx \leq 1\}$. Here, $B \coloneqq B(a,r)$ denotes the open ball in $\mathbb{R}^n$ centered at $a \in \mathbb{R}^n$ with radius $r > 0$, and $|B(a,r)|$ for its Lebesgue measure.

Meanwhile, for Θ is a Young function and $\theta \in G_\theta$, *the weak Orlicz–Morrey space* $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\| f \|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n, r>0} \frac{\Theta^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta\left(|B(a,r)|^{\frac{1}{n}}\right)} \| f \|_{wL_\Theta(B)} < \infty,$$

where

$$\| f \|_{wL_\Theta(B)} := \inf\left\{b > 0: \sup_{t>0}\Phi(t)\left|\left\{x \in B: \frac{|f(x)|}{b} > t\right\}\right| \leq 1\right\}.$$

The space $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ is quasi-Banach spaces equipped with the quasi-norm $\|\cdot\|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)}$. Note that, analog with $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ space, $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ also covers many classical spaces, which shown in the following example.

Example 1.1 *Let* $1 \leq p \leq q < \infty$, *Φ be a Young function, and* $\theta \in G_\theta$ *then we obtain:*

1. If $\Theta(t) := t^p$ and $\theta(t) := t^{\frac{-n}{p}}$, then $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) = wL^p(\mathbb{R}^n)$ is weak Lebesgue space.
2. If $\Theta(t) := t^q$ and $\theta(t) := t^{\frac{-n}{p}}$, then $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) = w\mathcal{M}_p^q(\mathbb{R}^n)$ is classical weak Morrey space.
3. If $\Theta(t) := t^p$, then $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) = w\mathcal{M}_\theta^p(\mathbb{R}^n)$ is generalized weak Morrey space.
4. If $\theta(t) := \Theta^{-1}(t^{-n})$, then $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) = wL_\Theta(\mathbb{R}^n)$ is weak Orlicz space.

Moreover, the relationship between $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ space and $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ space can be stated as the following lemma.

Lemma 1.2 *Let Θ be a Young function and* $\theta \in G_\Theta$. *Then* $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n) \subseteq w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ *with*

$$\| f \|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} \leq \| f \|_{\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)}$$

for every $f \in \mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$.

Many authors have been culminating important observations about inclusion properties of function spaces, see (Jiménez-Vargas *et al*., 2018; Maligranda and Matsuoka, 2015; Masta *et al*., 2018; Masta, 2018; Taqiyuddin and Masta, 2018; Diening, and Růžička, 2007), *etc*. Reports in literature (Masta *et al*., 2018) obtained sufficient and necessary conditions for inclusion of strong Orlicz–Morrey spaces of all versions. In the same paper , Masta *et al. (2017)* also proved the sufficient and necessary conditions for inclusion properties of weak Orlicz –Morrey spaces of Nakai 's and Sawano –Sugano – Tanaka's versions.

In this paper, we would like to obtain the inclusion properties of weak Orlicz–Morrey space $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ of Deringoz–Guliyev–Samko's version, and compare it with the result for Nakai's and Sawano-Sugano-Tanaka's versions.

2. METHODS

To obtain the sufficient and necessary conditions for inclusion properties of $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$, we used the similar methods in (Gunawan *et al*., 2017 , 2018 ; Masta *et al*., 2018 ; Masta , 2018 ; Osançlıol , 2014), which pay attention to the characteristic functions of open balls in $\mathbb{R}^n$, in the following lemma.

Lemma 1.3 *(Guliyev et al., 2017) Let Θ be a Young function, $\theta \in G_\theta$, and $r_0 > 0$, then there exists a constant $C > 0$ such that*
$$\frac{1}{\theta(r_0)} \leq \| \chi_{B(0,r_0)} \|_{\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} \leq \frac{C}{\theta(r_0)}.$$

For weak Orlicz–Morrey spaces, we have the following lemma.

Lemma 1.4 *Let Θ be a Young function, $\theta \in G_\theta$, and $r_0 > 0$, then there exists a constant $C > 0$ such that*
$$\frac{1}{\theta(r_0)} \leq \| \chi_{B(0,r_0)} \|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} \leq \frac{C}{\theta(r_0)}.$$

Proof. Since Θ is a Young function and $\theta \in G_\theta$, by Lemmas 1.2 and 1.3, we have

$$\| \chi_{B(0,r_0)} \|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} \leq \| \chi_{B(0,r_0)} \|_{\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)}$$
$$\leq \frac{C}{\theta(r_0)}.$$

On the other hand,

$$\| \chi_{B(0,r_0)} \|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} = \sup_{a\in\mathbb{R}^n, r>0} \frac{\Theta^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta(|B(a,r)|)} \| \chi_{B(0,r_0)} \|_{wL_{B(a,r)}}$$
$$= \sup_{a\in\mathbb{R}^n, r>0} \frac{\Theta^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta(|B(a,r)|^{\frac{1}{n}})} \frac{1}{\Theta^{-1}\left(\frac{|B(a,r)|}{|B(a,r)\cap B(0,r_0)|}\right)}$$
$$\geq \frac{1}{\theta(r_0)}.$$

Consequently, we have
$$\frac{1}{\theta(r_0)} \leq \| \chi_{B(0,r_0)} \|_{w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} \leq \frac{C}{\theta(r_0)}.$$

In this paper, the letter C was used for constants that may change from line to line, while constants with subscripts, such as C_1, C_2, do not change in different lines.

3. RESULTS AND DISCUSSION

First, we reproved sufficient and necessary conditions for inclusion properties of Orlicz–Morrey space $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ in the following theorem.

Teorema 2.1 *(Masta et al., 2018) Let Θ_1, Θ_2 be Young functions such that $\Theta_1 \prec \Theta_2$, $\Theta_1^{-1} \lesssim \Theta_2^{-1}$, $\theta_1 \in G_{\theta_1}$ and $\theta_2 \in G_{\theta_2}$. Then the following statements are equivalent:*

(1) $\theta_2 \lesssim \theta_1$.
(2) $\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq \mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)$.
(3) There exists a constant $C > 0$ such that
$$\| f \|_{\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)} \leq C \| f \|_{\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)}$$
for every $f \in \mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)$.

Proof. Assume that (1) holds and let $f \in \mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)$. Since $\Theta_1 \prec \Theta_2$, by using similar arguments in the proof of Corollary 2.3 in (Masta *et al*., 2016), we have

$$\| f \|_{L_{\Theta_1}(B(a,r))} \leq C \| f \|_{L_{\Theta_2}(B(a,r))}$$

for every $B(a,r) \subseteq \mathbb{R}^n$.

Knowing that, $\Theta_1^{-1} \lesssim \Theta_2^{-1}$ and $\theta_1 \lesssim \theta_2$ (i.e. there exists constant $C_1, C_2 > 0$ such that $\Theta_1^{-1}(t) \leq C_1\Theta_2^{-1}(t)$ and $\theta_2(t) \leq C_2\theta_1(t)$ for every $t > 0$), we obtain

$$\| f \|_{\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n, r>0} \frac{\Theta_1^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta_1\left(|B(a,r)|^{\frac{1}{n}}\right)} \| f \|_{L_{\Theta_1}(B(a,r))}$$
$$\leq \sup_{a\in\mathbb{R}^n, r>0} \frac{C\Theta_1^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta_1\left(|B(a,r)|^{\frac{1}{n}}\right)} \| f \|_{L_{\Theta_2}(B(a,r))}$$
$$\leq \sup_{a\in\mathbb{R}^n, r>0} \frac{CC_1\Theta_2^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta_1(|B(a,r)|^{\frac{1}{n}})} \| f \|_{L_{\Theta_2}(B(a,r))}$$

$$\leq \sup_{a\in\mathbb{R}^n, r>0} \frac{CC_1C_2\Theta_2^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta_2\left(|B(a,r)|^{\frac{1}{n}}\right)} \parallel f \parallel_{L_{\Theta_2}(B(a,r))}$$

$$:= CC_1C_2 \parallel f \parallel_{\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)}.$$

This proves that $\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq \mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)$.

Next, since $(\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n), \mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n))$ is a Banach pair, it follows from Lemma 3.3 that (2) and (3) are equivalent . It thus remains to show that (3) implies (1).

Assuming that (3) holds. Let $r_0 > 0$ and adding Lemma 1.3, we have

$$\frac{1}{\theta_1(r_0)} \leq \parallel \chi_{B(0,r_0)} \parallel_{\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)} \leq C \parallel \chi_{B(0,r_0)} \parallel_{\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)} \leq \frac{C}{\theta_2(r_0)},$$

Since $r_0 > 0$ is arbitrary, we conclude that $\theta_2(t) \leq C\theta_1(t)$ for every $t > 0$.

Now we come to the inclusion property of weak Orlicz–Morrey spaces $w\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)$ and $w\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)$ with respect to Young functions Θ_1, Θ_2 and parameters θ_1, θ_2.

Teorema 2.2 *Let Θ_1, Θ_2 be Young functions such that $\Theta_1 \prec \Theta_2$, $\Theta_1^{-1} \lesssim \Theta_2^{-1}$, $\theta_1 \in G_{\theta_1}$ and $\theta_2 \in G_{\theta_2}$. Then the following statements are equivalent:*

(1) $\theta_2 \lesssim \theta_1$.
(2) $w\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq w\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)$.
(3) There exists a constant $C > 0$ such that $\parallel f \parallel_{w\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)} \leq C \parallel f \parallel_{w\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)}$

for every $f \in w\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)$.

Proof. Assume that (1) holds and let $f \in w\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)$. Since $\Theta_1 \prec \Theta_2$, by using similar arguments in the proof of Theorem 3.3 in Masta et al., 2016, we have

$$\parallel f \parallel_{wL_{\Theta_1}(B(a,r))} \leq C \parallel f \parallel_{wL_{\Theta_2}(B(a,r))}$$

for every $B(a,r) \subseteq \mathbb{R}^n$.

Knowing that, $\Theta_1^{-1} \lesssim \Theta_2^{-1}$ and $\theta_1 \lesssim \theta_2$ (i.e there exists constant $C_1, C_2 > 0$ such that $\Theta_1^{-1}(t) \leq C_1\Theta_2^{-1}(t)$ and $\theta_2(t) \leq C_2\theta_1(t)$ for every $t > 0$), we obtain

$$\parallel f \parallel_{w\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n, r>0} \frac{\Theta_1^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta_1\left(|B(a,r)|^{\frac{1}{n}}\right)} \parallel f \parallel_{wL_{\Theta_1}(B(a,r))}$$

$$\leq \sup_{a\in\mathbb{R}^n, r>0} \frac{C\Theta_1^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta_1\left(|B(a,r)|^{\frac{1}{n}}\right)} \parallel f \parallel_{wL_{\Theta_2}(B(a,r))}$$

$$\leq \sup_{a\in\mathbb{R}^n, r>0} \frac{CC_1\Theta_2^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta_1\left(|B(a,r)|^{\frac{1}{n}}\right)} \parallel f \parallel_{wL_{\Theta_2}(B(a,r))}$$

$$\leq \sup_{a\in\mathbb{R}^n, r>0} \frac{CC_1C_2\Theta_2^{-1}\left(\frac{1}{|B(a,r)|}\right)}{\theta_2\left(|B(a,r)|^{\frac{1}{n}}\right)} \parallel f \parallel_{wL_{\Theta_2}(B(a,r))}$$

$$:= CC_1C_2 \parallel f \parallel_{w\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)}.$$

This proves that $w\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq w\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)$.

As a corollary of the Open Mapping Theorem (Appendix G in Grafakos *et al.*, 2019), we are aware that Chapter I, Lemma 3.3 still holds for quasi-Banach spaces, and so (2) and (3) are equivalent.

Now, assume that (3) holds. Let $r_0 > 0$. By Lemma 1.4, we have

$$\frac{1}{\theta_1(r_0)} \leq \parallel \chi_{B(0,r_0)} \parallel_{w\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)} \leq C \parallel \chi_{B(0,r_0)} \parallel_{w\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)} \leq \frac{C}{\theta_2(r_0)},$$

Since $r_0 > 0$ is arbitrary, we conclude that $\theta_2(t) \leq C\theta_1(t)$ for every $t > 0$.

For generalized weak Morrey spaces, we have the following corollary.

Corollary 2.3 *Let $1 \leq p < \infty$, $\theta_1 \in G_{\theta_1}$ and $\theta_2 \in G_{\theta_2}$. Then the following statements are equivalent:*

(1) $\theta_2 \lesssim \theta_1$.
(2) $w\mathcal{M}^p_{\theta_2}(\mathbb{R}^n) \subseteq w\mathcal{M}^p_{\theta_1}(\mathbb{R}^n)$.
(3) There exists a constant $C > 0$ such that $\parallel f \parallel_{w\mathcal{M}^p_{\theta_1}(\mathbb{R}^n)} \leq C \parallel f \parallel_{w\mathcal{M}^p_{\theta_2}(\mathbb{R}^n)}$ for every $f \in w\mathcal{M}^p_{\theta_2}(\mathbb{R}^n)$.

We have shown the sufficient and necessary conditions for the inclusion relation between weak Orlicz–Morrey space $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$. In the proof of the inclusion property, we used the norm of characteristic function on $\mathbb{R}^n$. The inclusion properties of weak Orlicz-Morrey space $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ (Theorem 2.2) and weak Orlicz–Morrey space $w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ of Sawano–Sugano–Tanaka's version (Theorem 3.9 in previous report (Masta *et al*., 2018)) generalized the inclusion properties of weak Morrey spaces and resulted weak Morrey spaces in literature (Masta *et al*., 2017a; 2017b). Meanwhile, the inclusion properties of weak Orlicz-Morrey space $wL_{\phi,\Phi}(\mathbb{R}^n)$ of Nakai's version (Theorem 3.4 in literature (Masta *et al*., 2018) generalized the unique inclusion properties of weak Orlicz spaces in other report (Masta *et al*., 2016).

Comparing Theorem 2.2 and Theorem 3.9 in Masta *et al*. (2018), we say that the condition on the Young function for the inclusion of the weak Orlicz–Morrey space $w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ is simpler than that for the weak Orlicz–Morrey space $w\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$.

As a corollary of Lemma 1.2, Theorem 2.1 and Theorem 2.2, we also have the following inclusion relations

$$\begin{array}{ccc} \mathcal{M}_{\theta_2,\Theta_2} & \rightarrow & \mathcal{M}_{\theta_1,\Theta_1} \\ \downarrow & \searrow & \downarrow \\ w\mathcal{M}_{\theta_2,\Theta_2} & \rightarrow & w\mathcal{M}_{\theta_1,\Theta_1} \end{array}$$

for $\Theta_1 \prec \Theta_2$, $\Theta_1^{-1} \lesssim \Theta_2^{-1}$ and $\theta_2 \lesssim \theta_1$, where the arrows mean 'contained in' or 'embedded into'.

4. CONCLUSION

This article has discussed the third version of weak Orlicz–Morrey space, which is an enlargement of third version of (strong) Orlicz–Morrey space. This study also investigated some properties of the third version of weak Orlicz–Morrey spaces, especially the sufficient and necessary conditions for inclusion relations between two these spaces. One of the keys to get our result is to estimate the quasi-norm of characteristics function of open balls in $\mathbb{R}^n$.

5. AUTHORS' NOTE

The author(s) declare(s) that there is no conflict of interest regarding the publication of this article. Authors confirmed that the data and the paper are free of plagiarism.

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