HEAT DIFFUSION IN TWO-DIMENSIONAL MEDIA CONTAINING SMALL-SCALE CRACK

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ABSTRAK

Dalam artikel ini, kita mempelajari difusi panas dalam media berdimensi dua yang memuat crack. Ada dua jenis crack yang kita pandang, crack lentur dan crack keras. Kedua jenis crack itu dibedakan oleh syarat batas dari persamaan difusi. Dengan menggunakan metode dari Muijres, kita dapat menentukan temperatur di luar crack yang dinyatakan dalam fungsi Green dari persamaan difusi dan temperatur incident pada crack.

Key words: *crack, diffusion, incident field, scattered field*

INTRODUCTION

The problem with small scale heterogeneities is, that they cannot be observed individually using seismic waves, but still can have a significant effect on the amplitude and phase of the transmitted wave field. Due to the small scale of the cracks, the medium almost behaves as a homogeneous one. Most methods concerning wave propagation in cracked media are based on this concept on an effective medium.

Muijres (1998) proposed a method for solving the boundary-value problem corresponding to waves propagating through a two-dimensional medium containing a large number of small-scale inhomogeneities. The method was applied to three types of scattering objects in a homogeneous and unbounded embedding: (1) compliant cracks accounted for by the Dirichlet boundary condition, (2) rigid cracks accounted for by the Neumann boundary condition and (3) penetrable heterogeneity characterized by a compressibility that differs from the compressibility in the embedding.

Van Baren (1998) constructed a finite difference method, capable of solving the scattering problem in case of a large number of small-scale cracks embedded in a heterogeneous medium. This method accounts for the presence of the cracks by introducing secondary point source instead of imposing explicit boundary conditions. The reduction in computing-time compared to the method of Muijres is significant.

In this paper we study the diffusion of heat through a two-dimensional medium containing a crack. Our method is similar to the method described by Muijres (1998) in case of acoustic waves. We concentrate on a compliant crack and a rigid crack. Starting from an integral representation for the temperature, an integral equation is obtained for the unknown field quantity at the crack. By choosing adequate expansion function, we determine the unknown expansion coefficient.

FORMULATION PROBLEM

We consider two-dimensional scattering from a crack embedded in a homogeneous medium, for heat diffusion. The thermal diffusivity of the medium, α , is a constant. The crack is denoted as a line segment Σ characterized by its position $\mathbf{x_0} = (x_0, z_0)$, its half-width a and the angle φ with horizontal. The z-coordinate indicates depth, whereas the x-coordinate refers to the horizontal position. The unity vector normal to the crack is given by $\mathbf{n} = (\cos\varphi, \sin\varphi)$.

The total temperature field u is written as a superposition of the incident field u^{inc} and the scattered field u^{sc}:

$$u(\mathbf{x}, \omega) = u^{inc}(\mathbf{x}, \omega) + u^{sc}(\mathbf{x}, \omega), \qquad (1)$$

where $\mathbf{x} = (x, z)$ is a Cartesian position vector and ω is the angular frequency. Our formulation is in the temporal frequency domain, but for brevity we omit the explicit ω-dependence in the remainder.

In the region outside the cracks, the temperature satisfies the diffusion equation

$$\nabla^2 u(\mathbf{x}) + \frac{\sqrt{i\omega}}{\alpha} u(\mathbf{x}) = -s(\mathbf{x}), \qquad (2)$$

where α is the thermal diffusivity associated with the speed propagation of heat in the medium during changes of the temperature with time, and $s(\mathbf{x})$ represents the source that generates the incident field. In case of a unit point source located in $\mathbf{x} = \mathbf{x}'$, that is , $s(\mathrm{x}) = \delta(\mathbf{x} - \mathbf{x}')$ the solution of Eq. (2) corresponding to outgoing temperature is Green's function u^G, for the background medium. This reads

$$u^G(\mathbf{x}, \mathbf{x}') = \frac{i}{4} H_0^{(1)}(k_0 r) \qquad (3)$$

where i is the imaginary unit, $H_0^{(1)}$ is the zeroth-order Hankel function of the first kind, k_0 is the diffusion number ($k_0 = \sqrt{i\omega/\alpha}$) and r is the distance between $\mathbf{x}$ and $\mathbf{x'}$

$$r(\mathbf{x},\mathbf{x}') = |\mathbf{x}-\mathbf{x}'| \quad (4)$$

$$= \sqrt{(x-x')^2+(y-y')^2} \quad (5)$$

The following integral representation can be derived for the temperature outside the crack:

$$u(\mathbf{x}) = u^{inc}(\mathbf{x}) - \int_{x'\in\Sigma} d\mathbf{x}' G(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}'), \qquad \mathbf{x}\notin\Sigma \quad (6)$$

where G follow from the two-dimensional Green's function (3), and ϕ depend on the particular scattering problem considered. In order to determine the unknown quantity ϕ at the crack a Fredholm integral equation has to be derived from (6) by letting the point observation approach the crack and using the boundary condition:

$$\lim_{x\to xo} q^{inc}(\mathbf{x}) = \lim_{x\to x_o} \int_{x'\in\Sigma} d\mathbf{x}' K(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}') \quad (7)$$

The precise form of the Kernel K (determined by Green's function u^G) and the function q^{inc} (determined by the incident field u^{inc}) depends on the type of the cracks; expression for the compliant crack and the rigid crack are given in following sections.

DISCRETIZATION

In order to evaluate the integrals in Eq. (6) and (7), ϕ is expanded in terms of an appropriately chosen sequence of functions. Following the work of Muijres (1998) but for arbitrary φ we split up ϕ in a frequency-dependent and a spatial part,

$$\phi(\mathbf{x},\omega) = b(\omega)\psi(\mathbf{x}), \qquad \mathbf{x}\in\Sigma \quad (8)$$

with ψ a suitable expansion function chosen to be independent of frequency, the frequency dependence resides in *b*.

After multiplying Eq. (7) with a conveniently chosen weight function $w(\mathbf{x})$, and integrating the result over the crack we finally arrive at

$$q^{inc} = K_{11}b \quad (9)$$

with

$$q^{inc} = \int_{x\in\Sigma} d\mathbf{x}\; w(\mathbf{x})\; q^{inc}(\mathbf{x}) \quad (9a)$$

and

$$K_{11} = \int_{x\in\Sigma} d\mathbf{x}\, \mathrm{w}(\mathbf{x}) \int_{x'\in\Sigma} d\mathbf{x}'\, K(\mathbf{a}\mathrm{x}')\psi(\mathbf{x}') \tag{9b}$$

Esq. (9) has to be solved for the coefficient b. Subsequently, the temperature outside the crack can be computed by substituting Eq. (8) in Eq. (6):

$$u(\mathbf{x}) = u^{inc}(\mathbf{x}) - b \int_{x'\in\Sigma} d\mathbf{x}'\, G(\mathbf{x}, \mathbf{x}')\psi(\mathbf{x}'), \qquad \mathbf{x} \notin \Sigma \tag{10}$$

We apply our framework to two type of cracks differing in boundary conditions: vanishing temperature at a compliant crack (Dirichlet boundary condition) and vanishing component of the temperature gradient at a rigid crack (Neumann boundary condition). The two cases are discussed below in more detail.

COMPLIANT CRACK

To simplify the following equations, we introduce a new coordinate system (η, ξ) such that the position of the crack can be simplified to be $(0,0)$ in new coordinate system.

The compliant crack is characterized by the Dirichlet boundary condition:

$$u(\eta, \xi = 0) = 0, \qquad |\eta| < a \tag{11}$$

The function ϕ represents the jump in normal derivative of the temperature across the crack:

$$\phi(\eta) = \lim_{\xi\downarrow 0} \frac{\partial u}{\partial \xi}(\eta, \xi) - \lim_{\xi\uparrow 0} \frac{\partial u}{\partial \xi}(\eta, \xi), \qquad |\eta| < a \tag{12}$$

and G is Green' function (3) for the embedding medium

$$G(\eta, \xi; \eta', \xi') = u^{G}(\eta, \xi; \eta', \xi') \tag{13}$$

To obtain a Fredholm equation of the first kind for the crack, we let the point of observation approach the crack and use boundary condition (11). We get

$$u^{inc}(\eta, \xi = 0) = \int_{-a}^{a} d\eta'\, u^{G}(\eta, \xi = 0; \eta', \xi' = 0)\phi(\eta'), \qquad |\eta| < a \tag{14}$$

The kernel of this integral equation is Green's function (3):

$$K(\eta, \xi; \eta', \xi') = u^{G}(\eta, \xi; \eta', \xi'), \tag{15}$$

furthermore, the function q^{inc} is the incident field:

$$q^{inc}(\eta, \xi) = u^{inc}(\eta, \xi) \tag{16}$$

For both expansion and weight function we make the following choice:

$$\psi(\eta) = \{a^2 - \eta^2\}^{-1/2}, \qquad w(\eta) = \{a^2 - \eta^2\}^{-1/2}. \tag{17}$$

This choice is based on a solution for a small crack (see Muijres (1998)). To calculate q^{inc} in Eq. (9a), we expand u^{inc} in the zeroth-order Taylor series and obtain

$$q^{inc} = \pi u^{inc}(0,0). \tag{18}$$

To calculate K_{11} we approximate Green function $H_0^{(1)}(k_0 r)$ by $\log(k_0 r)$ (because the argument approach zero). Integrating Eq. (9b) we obtain

$$K_{11} = \frac{\pi^2 i}{4} \tag{19}$$

For the temperature outside the crack we derive from Eq. (10) the following expression:

$$u(\mathbf{x}) = u^{inc}(\mathbf{x}) - H_0^{(1)}(k_0 r(\mathbf{x},\mathbf{x}_o) q(\omega) \tag{20}$$

with $q(\omega) = u^{inc}(x_0, y_0)$.

RIGID CRACK

This is characterized by the Neumann boundary condition:

$$\lim_{\xi \to 0} = \frac{\partial u}{\partial \xi}(\eta, \xi) = 0, \qquad |\eta| < a \tag{21}$$

Now the function ϕ represents the jump in the temperature across the crack:

$$\phi(\eta) = \lim_{\xi \downarrow 0} u(\eta, \xi) - \lim_{\xi \uparrow 0} u(\eta, \xi), \qquad |\eta| < a, \tag{22}$$

and G is the partial derivative with respect to ξ' of Green's function (3):

$$G(\eta, \xi; \eta', \xi') = \frac{\partial u^G}{\partial \xi'}(\eta, \xi; \eta', \xi'). \tag{23}$$

To obtain a Fredholm equation of the first kind we first take the partial derivative of (6) with respect to ξ, we then let the point of observation subsequently approach the crack, while using the boundary condition (21):

$$\frac{\partial u^{inc}}{\partial \xi}(\eta, \xi = 0) = -\lim_{\xi \to 0} \int_{-a}^{a} \frac{\partial^2 u^G}{\partial \xi \partial \xi'}(\eta, \xi; \eta', \xi' = 0), \qquad |\eta| < a \tag{24}$$

When we compare this equation with Eq. (7) we see that K is the kernel

$$K(\eta,\xi;\eta',\xi') = \frac{\partial^2 u^G}{\partial\xi\partial\xi'}(\eta,\xi;\eta',\xi') \tag{25}$$

and that q^{inc} is the function

$$q^{inc}(\eta,\xi) = \lim_{\xi\to 0}\frac{\partial u^{inc}}{\partial\xi}(\eta,\xi). \tag{26}$$

For both expansion and weight function we make the following choice:

$$\psi(\eta) = \sqrt{2}\sqrt{a^2-\eta^2}, \qquad w(\eta) = \sqrt{2}\sqrt{a^2-\eta^2}\ . \tag{27}$$

Again this choice is based on a solution for a small crack (see also Muijres (1998)). To calculate q^{inc} in Eq. (9a), we expand $\partial u^{inc}/\partial\xi$ in a first-order Taylor series and obtain

$$q^{inc} = \frac{\pi a^2}{\sqrt{2}}\frac{\partial u^{inc}}{\partial\xi}(\eta=0,\xi=0). \tag{28}$$

To calculate K_{11} we again approximate Green function $H_0^{(1)}(k_0 r)$ by $\log(k_0 r)$. Integrating Eq. (9b) we obtain

$$K_{11} = \frac{\pi a^2}{2} \tag{29}$$

For the temperature outside the crack we derive from Eq. (10) the following expression:

$$u(\mathbf{x}) = u^{inc}(\mathbf{x}) + \pi a^2\frac{n\cdot(x-x_0)}{r(x,x_0)}k_0\frac{i}{4}H_1^{(1)}(k_0 r(\mathbf{x},\mathbf{x}_0))q(\omega) \tag{30}$$

with $q(\omega) = \mathbf{n}\cdot\nabla u^{inc}(x_0,y_0)$

CONCLUSIONS

We apply the method of Muijres for solving the boundary-value problem corresponding to diffusion through a two-dimensional medium containing a crack. We consider two type of crack: (1) compliant crack and (2) rigid crack. Based on the method, we derive the integral representation for the temperature field outside the crack. By using the numerical techniques to solve unknown expansion coefficient, we are able to consider diffusion through media containing a crack..

APPENDIX

The Green's function is a solution for the diffusion equation with source point at $\mathbf{x}' = (x_o, z_o)$:

$$\nabla^2 u(x,x',t) - \frac{1}{\alpha^2}\frac{\partial}{\partial t}u(x,x',t) = -\delta(x-x_o)(z-z_o)\delta(t), \qquad (A.1)$$

where $\mathbf{x} = (x,z)$.

There are several ways to obtain expression for the Green's function. One way is by performing Fourier transform in space. Let the inverse formula for Fourier transform of the Green's function, u^G be denoted by γ. Then

$$u^G(x,x',t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda\cdot(x-x')}\gamma(\lambda,t)d\lambda_1 d\lambda_2 \qquad (A.2)$$

with $\lambda = (\lambda_1, \lambda_2)$. Since

$$\nabla^2 u^G - \frac{1}{\alpha^2}\frac{\partial}{\partial t}u^G = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda\cdot(x-x')}\left[-\lambda^2\gamma - \frac{1}{\alpha^2}\left(\frac{\partial\gamma}{\partial t}\right)\right]d\lambda_1 d\lambda_2 \qquad (A.3)$$

with $\lambda^2 = \lambda_1^2 + \lambda_2^2$, and

$$\delta(x-x_o)\delta(z-z_o) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda\cdot(x-x')}d\lambda_1 d\lambda_2 \qquad (A.4)$$

We finally obtain an equation for γ:

$$\frac{1}{\alpha^2}\left(\frac{\partial\gamma}{\partial t}\right) + \lambda^2\gamma = \delta(t) \qquad (A.5)$$

with a solution

$$\gamma(\lambda,t) = \alpha^2 e^{-\alpha^2\lambda^2 t}H(t) \qquad (A.6)$$

Substituting (A.6) in (A.2) we obtain

$$u^G(x,x',t) = \frac{\alpha^2 H(t)}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda\cdot(x-x')}e^{-\alpha^2\lambda^2 t}d\lambda_1 d\lambda_2 \qquad (A.7)$$

in which $H(t)$ is the Heaviside step function.

We can evaluate this integral explicitly by completing the square of exponential argument.

$$-i\lambda\cdot(x-x') - \alpha^2\lambda^2 t = -\left(\alpha\sqrt{t}\lambda + \frac{i(x-x')}{2\alpha\sqrt{t}}\right)^2 - \left(\frac{|x-x'|^2}{4\alpha^2 t}\right)$$

$$= -\alpha^2 ts^2 - \frac{|x-x'|^2}{4\alpha^2 t}$$

Hence the integral may be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\alpha^2 t)s^2-\frac{|x-x'|^2}{4\alpha^2 t}}\, d\lambda_1 d\lambda_2; \qquad s=\lambda+\frac{i(x-x')}{2\alpha^2\sqrt{t}} \tag{A.8}$$

and by an appropriate change of variable this equals

$$e^{-\frac{|x-x'|^2}{4\alpha^2 t}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\alpha^2 t)s^2}\, ds_1 ds_2 = \frac{\pi}{\alpha^2 t} e^{-\frac{|x-x'|^2}{4\alpha^2 t}} \tag{A.9}$$

Introducing this result into the expression for u^G we obtain

$$u^G(x,x',t)=\frac{e^{-\frac{|x-x'|^2}{4\alpha^2 t}}}{4\pi t}H(t) \tag{A.10}$$

Other form of the Green's function can be derive from the diffusion equation in frequency domain and we obtain (see Morse & Freshbach (1953))

$$u^G(x,x',\omega)=\frac{i}{4}H_0^{(1)}(k_0 r) \tag{A.11}$$

where $H_0^{(1)}(x)$ is the Hankel function of first kind, zeroth-order.

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